

On the Eigenvalue Spacing Distribution for a Point Scatterer on the Flat Torus

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Abstract. We study the level spacing distribution for the spectrum of a point scatterer on a flat torus. In the two-dimensional case, we show that in the weak coupling regime, the eigenvalue spacing distribution coincides with that of the spectrum of the Laplacian (ignoring multiplicities), by showing that the perturbed eigenvalues generically clump with the unperturbed ones on the scale of the mean level spacing. We also study the three dimensional case, where the situation is very different.

1. Introduction

1.1. The Šeba Billiard

Point scatterers are toy models used to understand aspects of quantum systems for which the corresponding classical limit is intermediate between integrable and chaotic. In this paper, we study the spectral statistics of point scatterers on the flat torus (a “Šeba billiard”) in the “weak coupling” regime.

A point scatterer on the torus is formally given by a Hamiltonian

$$-\Delta + \alpha\delta_{x_0}, \quad \alpha \in \mathbb{R} \quad (1.1)$$

where Δ is the Laplacian, α denotes a coupling constant and x_0 denotes the position of the scatterer. Mathematically, a point scatterer is realised as a self-adjoint extension of the Laplacian $-\Delta$ acting on functions which vanish near x_0 (see [3]). Such extensions are parameterized by a phase $\varphi \in (-\pi, \pi]$, where $\varphi = \pi$ corresponds to the standard Laplacian ($\alpha = 0$ in (1.1)). We denote the corresponding operator by $-\Delta_{x_0, \varphi}$. For $\varphi = \pi$, the eigenvalues are those of the standard Laplacian. For $\varphi \neq \pi$ ($\alpha \neq 0$), the resulting spectral problem still has the eigenvalues of the unperturbed problem, with multiplicity decreased by one, as well as a new set of $\Lambda_\varphi = \{\lambda_j^\varphi\}$ of eigenvalues interlaced between

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the sequence of unperturbed eigenvalues, each appearing with multiplicity one, and satisfying the spectral equation

$$\sum_{j=0}^{\infty} |\psi_j(x_0)|^2 \left(\frac{1}{\lambda_j - \lambda} - \frac{\lambda_j}{\lambda_j^2 + 1} \right) = c_0 \tan \frac{\varphi}{2} \tag{1.2}$$

where $c_0 = \sum_{j=0}^{\infty} \frac{|\psi_j(x_0)|^2}{\lambda_j^2 + 1}$ and $\{\psi_j\}$ form an orthonormal basis of eigenfunctions for the unperturbed problem: $-\Delta \psi_j = \lambda_j \psi_j$. The eigenfunction corresponding to $\lambda \in \Lambda_\varphi$ is the Green's function $G_\lambda(x, x_0) = (\Delta + \lambda)^{-1} \delta_{x_0}$.

We denote the unperturbed eigenvalues without multiplicities by

$$\mathcal{N} = \{n_0 = 0 < n_1 < \dots < n_j < \dots\}$$

and call them the “norms” of the torus. Note that the perturbed eigenvalues defined by (1.2) are independent of the location x_0 of the scatterer, since in the case of the torus, the sums $\sum_{\lambda_j=n} |\psi_j(x_0)|^2 = \#\{\lambda_j = n\}$ are independent of x_0 .

1.2. Spacing Distributions

The perturbed eigenvalues $\{\lambda_j^\varphi\}$ interlace with the norms $\{n_j\}$ as follows

$$\lambda_0^\varphi < 0 = n_0 < \lambda_1^\varphi < n_1 < \dots < \lambda_k^\varphi < n_k < \dots \tag{1.3}$$

The nearest neighbour spacings for the norms and for the perturbed eigenvalues are defined by

$$\delta_j := n_{j+1} - n_j, \quad \delta_j^\varphi := \lambda_{j+1}^\varphi - \lambda_j^\varphi \tag{1.4}$$

The *mean spacing* between the norms is defined by

$$\langle \delta_j \rangle_x := \frac{1}{N(x)} \sum_{n_j \leq x} \delta_j \sim \frac{x}{N(x)}, \quad x \rightarrow \infty \tag{1.5}$$

where

$$N(x) := \#\{j : n_j \leq x\} \tag{1.6}$$

and likewise for the mean spacing $\langle \delta_j^\varphi \rangle_x$ between the new eigenvalues. Clearly

$$\langle \delta_j^\varphi \rangle_x \sim \langle \delta_j \rangle_x, \quad x \rightarrow \infty \tag{1.7}$$

We define normalised nearest neighbour spacings by

$$\widehat{\delta}_j := \frac{\delta_j}{\langle \delta_j \rangle_x}, \quad \widehat{\delta}_j^\varphi := \frac{\delta_j^\varphi}{\langle \delta_j^\varphi \rangle_x} \tag{1.8}$$

We want to determine the distribution of the normalised spacings δ_j^φ .

Shigehara et al. [9,10,12] identify two regimes in the semiclassical limit for a point scatterer in dimension 2: In the weak coupling regime, the phase φ is fixed as $\lambda \rightarrow \infty$. In this regime, the authors predict a Poissonian level spacing distribution for the perturbed spectrum. The strong coupling regime is when φ varies as $\lambda \rightarrow \infty$ so as to satisfy: $c_0 \tan \frac{\varphi}{2} \sim -\frac{1}{4\pi} \log \lambda$. where they predict level repulsion. In most numerical studies of this problem, it is the

second regime that appears, due to a truncation procedure [8]. For an analytic study of this regime, see [2, 6, 13].

We deal with the spectrum of a point scatterer in the weak coupling regime (φ fixed). We will show that the level spacing distributions of the norms and of the perturbed spectrum (if either exists) coincide. Since it is generally believed that the spacing distribution of the norms is Poissonian (if the torus is either rational, such as the standard torus $\mathbb{R}^2/\mathbb{Z}^2$, or generic irrational in a suitable sense [4]), that would imply that the perturbed spectrum is also Poissonian.

1.3. Our Results

We denote the differences between the old and new eigenvalues by

$$d_j := n_j - \lambda_j^\varphi > 0 \quad (1.9)$$

Since $\delta_j - \delta_j^\varphi = d_{j+1} - d_j$, the normalised nearest neighbour spacings between the norms and the perturbed spectrum are related by

$$\widehat{\delta}_j - \widehat{\delta}_j^\varphi \sim \frac{d_{j+1} - d_j}{\langle \delta_j \rangle_x} \quad (1.10)$$

We define the mean difference of d_j by

$$\langle d_j \rangle_x = \frac{1}{N(x)} \sum_{\lambda_j^\varphi \leq x} d_j. \quad (1.11)$$

We will show that the ratio between the mean difference d_j and the mean spacing δ_j vanishes:

Theorem 1.1. *For a point scatterer on a two-dimensional flat torus,*

$$\frac{\langle d_j \rangle_x}{\langle \delta_j \rangle_x} \rightarrow 0, \quad \text{as } x \rightarrow \infty. \quad (1.12)$$

As a consequence, since the differences $d_j \geq 0$ are non-negative, we deduce

Corollary 1.2. *Outside of a zero-density subsequence,*

$$\frac{d_j}{\langle \delta_j \rangle} \rightarrow 0, \quad \text{as } j \rightarrow \infty. \quad (1.13)$$

That is the norms and the perturbed eigenvalues clump together generically¹ on the scale of the mean spacing. Therefore,

Corollary 1.3. *If the spacings δ_j for the norms have a limiting distribution, then so do the spacings δ_j^φ for the perturbed spectrum and the limiting distributions coincide.*

A similar result holds for hyperbolic surfaces if the point scatterer is placed in a generic position. We will not give the details here.

¹ Recently Tudorovskiy et al. [14] presented a heuristic argument that in the fixed regime the spacing distribution should be Poissonian by claiming the bound (1.13) holds individually, for all j . We are unable to verify this.

1.4. Dimension 3

The situation is very different for a three-dimensional torus \mathbb{T}^3 . Let η_j^φ be the perturbed eigenvalues of the point scatterer and η_j the unperturbed eigenvalues counted without multiplicity (the norms). The ordering is

$$\eta_0^\varphi < 0 = \eta_0 < \eta_1^\varphi < \eta_1 < \cdots < \eta_j^\varphi < \eta_j \quad (1.14)$$

As before we let $d_j := \eta_j - \eta_j^\varphi$ and $\delta_j = \eta_{j+1} - \eta_j$. We denote by $\langle d_j \rangle_x$ the average of the spacings d_j , and by $\langle \delta_j \rangle_x$ of the spacings of the norms, for $\eta_j^\varphi \leq x$.

Theorem 1.4. *For the three-dimensional flat torus, we have*

$$\lim_{x \rightarrow \infty} \frac{\langle d_j \rangle_x}{\langle \delta_j \rangle_x} = \frac{1}{2}. \quad (1.15)$$

Note that Theorem 1.4 does not give any information on the relation between level spacing distributions for the norms and for the perturbed spectrum. For an empirical study of the spectral statistics in dimension 3, see [11].

2. Overview of the Proof

2.1. Our Method

We derive Theorems 1.1 and 1.4 from the asymptotics as $\beta \rightarrow 0$ of the sum

$$\sum_{j=0}^{\infty} d_j e^{-\beta \lambda_j^\varphi}$$

To so, we approximate the sum by the difference

$$\sum_{j=0}^{\infty} \{e^{-\beta \lambda_j^\varphi} - e^{-\beta n_j}\}$$

of the heat traces of the operators $-\Delta_{x_0, \varphi}$ and $-\Delta$, which we study in Sect. 5 by using a trace formula which will be developed in Sect. 3, 4 in the two-dimensional case. The three-dimensional case is treated in Sect. 6.

2.2. A Trace Formula for the Point Scatterer on the Torus

We work with a rectangular two-dimensional flat torus $\mathbb{T}^2 = \mathbb{R}^2 / 2\pi \mathcal{L}_0$, where $\mathcal{L}_0 = \mathbb{Z}(1/a, 0) \oplus \mathbb{Z}(0, a)$ for some $a > 0$. Denote by \mathcal{L} the dual lattice of \mathcal{L}_0 . The eigenvalues of the Laplacian on \mathbb{T}^2 are the norms of the vectors of the dual lattice \mathcal{L} (cf. section 2 in [7]). We denote the set of norms of the dual lattice vectors by

$$\mathcal{N} = \{0 < n_1 < \cdots\} \quad (2.1)$$

and the multiplicity of an eigenvalue $n \in \mathcal{N}$ is denoted by

$$r_{\mathcal{L}}(n) = \#\{\xi \in \mathcal{L} : |\xi|^2 = n\}. \quad (2.2)$$

Recall that the perturbed eigenvalues $\{\lambda_j^\varphi\}$ interlace with the norms $\{n_j\}$. The ordering is

$$\lambda_0^\varphi < 0 = n_0 < \lambda_1^\varphi < n_1 < \cdots < \lambda_j^\varphi < n_j. \quad (2.3)$$

(That $\lambda_0^\varphi < 0$ is given in [1]).

We denote $n_j = \rho_j^2$, where $\rho_j > 0$ for $j \geq 1$, and $\lambda_j^\varphi = (\rho_j^\varphi)^2$, where $\rho_j^\varphi > 0$ if $j \geq 1$ and $\Im \rho_0^\varphi > 0$ (note that $\lambda_0^\varphi < 0$ and ρ_0^φ is pure imaginary). The spectral function

$$S_\varphi(\rho) = -\frac{1}{\rho^2} + \sum_{j=1}^{\infty} r(n_j) \left\{ \frac{1}{n_j - \rho^2} - \frac{n_j}{n_j^2 + 1} \right\} - c_0 \tan \frac{\varphi}{2} \quad (2.4)$$

has simple poles at the points $\rho = \pm \rho_j$ and zeroes at the points $\rho = \pm \rho_j^\varphi$. For σ large enough and $\Im \rho = -\sigma$, we will show that

$$S_\varphi(\rho) = -\frac{1}{2\pi} \log(i\rho) + \frac{1}{2\pi} D(\rho) + c(\varphi) \quad (2.5)$$

for $c(\varphi) = c_1 - c_0 \tan \frac{\varphi}{2}$, where c_1 is some real constant, and $|D(\rho)| \ll_\sigma 1$.

Let h be an even function which is analytic in a strip $|\Im \rho| \leq \sigma'$ for some $\sigma' > \sigma$ and satisfies

$$|h(\rho)| \ll (1 + |\Re \rho|)^{-5-\delta}$$

for some $\delta > 0$ uniformly in the same strip. We have the following general trace formula which we prove in Sects. 3 and 4.

Let $\sigma > \sigma_0(\varphi)$ be sufficiently large. Then for all h as above, we have

$$\begin{aligned} \sum_{j=0}^{\infty} \{h(\rho_j^\varphi) - h(\rho_j)\} &= \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} \frac{h(\rho) d\rho}{\rho(\log i\rho - 2\pi c(\varphi))} \\ &\quad - \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} h'(\rho) \log \left(1 - \frac{D(\rho)}{\log i\rho - 2\pi c(\varphi)} \right) d\rho. \end{aligned} \quad (2.6)$$

2.3. A Tauberian Theorem

To prove Theorem 1.1, we will employ a Tauberian Theorem and reduce the problem to studying the asymptotics as $\beta \searrow 0$ of

$$\tilde{A}(\beta) = \sum_j d_j e^{-\beta \lambda_j^\varphi} \quad (2.7)$$

To study $\tilde{A}(\beta)$, we prove the following approximation (cf. (5.1) and Lemma 5.1)

$$\sum_{j=0}^{\infty} d_j e^{-\beta \lambda_j^\varphi} = \frac{1}{\beta} \sum_{j=0}^{\infty} \{e^{-\beta \lambda_j^\varphi} - e^{-\beta n_j}\} + O(\beta^{-1/2}). \quad (2.8)$$

We then use the trace formula (2.6) with $h(\rho) = e^{-\beta\rho^2}$ to bound $\sum_{j=0}^{\infty} \{e^{-\beta\lambda_j^{\varphi}} - e^{-\beta n_j}\}$ and obtain the following estimate which is the key result in the proof of Theorem 1.1:

Proposition 2.1. *As $\beta \searrow 0$,*

$$\tilde{A}(\beta) = \sum_j d_j e^{-\beta\lambda_j^{\varphi}} \ll \frac{1}{\beta \log \frac{1}{\beta}}. \tag{2.9}$$

2.4. Proof of Theorem 1.1

We will use Karamata’s Tauberian Theorem (see e.g. [5]) which deals with the following situation: We say a positive function $L(x)$ is *slowly varying* if $L(kt) \sim L(t)$ as $t \rightarrow \infty$ for each fixed $k > 0$. We are given a non-decreasing function $A(t)$ on \mathbb{R}_+ such that the Laplace transform

$$\tilde{A}(\beta) := \int_0^{\infty} e^{-\beta t} dA(t) \tag{2.10}$$

converges for all $\beta > 0$. Suppose there exists two real numbers $c \geq 0, \omega > 0$ and a slowly varying function $L(x)$ so that

$$\tilde{A}(\beta) = \{c + o(1)\} \beta^{-\omega} L(1/\beta), \quad \beta \searrow 0 \tag{2.11}$$

Then

$$A(x) = \{c + o(1)\} \frac{x^{\omega} L(x)}{\Gamma(\omega + 1)}, \quad x \rightarrow \infty \tag{2.12}$$

We apply Karamata’s Tauberian theorem to the function

$$A(x) := \sum_{\lambda_j \leq x} d_j \tag{2.13}$$

which is non-decreasing since $d_j \geq 0$. The Laplace transform \tilde{A} is

$$\tilde{A}(\beta) = \sum_j d_j e^{-\beta\lambda_j^{\varphi}}. \tag{2.14}$$

Proposition 2.1 implies that $\tilde{A}(\beta) = o(1/(\beta\sqrt{\log \frac{1}{\beta}}))$. Thus in Karamata’s theorem, we may take $\omega = 1, L(t) = 1/\sqrt{\log t}$, and $c = 0$ to find

$$A(x) = o\left(\frac{x}{\sqrt{\log x}}\right), \quad x \rightarrow \infty. \tag{2.15}$$

Therefore,

$$\frac{\langle d_j \rangle_x}{\langle \delta_j \rangle_x} = \frac{A(x) N(x)}{N(x) x} = \frac{A(x)}{x} = o\left(\frac{1}{\sqrt{\log x}}\right) \tag{2.16}$$

as $x \rightarrow \infty$, proving Theorem 1.1.

2.5. Three-Dimensional Tori

As in the two-dimensional case, Theorem 1.4 follows from the following proposition which we prove in Sect. 6.

Proposition 2.2. *We have as $\beta \searrow 0$*

$$\sum_{j=0}^{\infty} d_j e^{-\beta \eta_j^\varphi} = \frac{1}{2\beta} + O(\beta^{-3/4}). \quad (2.17)$$

The key tools in the derivation are a trace formula (cf. Theorem 6.3) and an approximation lemma (cf. Lemma 6.5).

3. The Trace Formula

We follow the same path as in [15] for a compact quotient $\Gamma \backslash \mathbb{H}$.

3.1. Overview of the Proof

Let $T > 0$ be such that $T \notin \{\rho_j\} \cup \{\rho_j^\varphi\}$, $\sigma > \Im \rho_0^\varphi$ and consider the box

$$B(\sigma, T) = \{\rho \mid |\Im \rho| \leq \sigma, |\Re \rho| \leq T\}.$$

For σ large enough and $\Im \rho = -\sigma$, we will show that the spectral function (2.4) can be written as

$$S_\varphi(\rho) = -\frac{1}{2\pi} \log(i\rho) + \frac{1}{2\pi} D(\rho) + c(\varphi) \quad (3.1)$$

for $c(\varphi) = c_1 - c_0 \tan \frac{\varphi}{2}$, where c_1 is some real constant, and $|D(\rho)| \ll_\sigma 1$.

Let h be an even function which is analytic in a strip $|\Im \rho| \leq \sigma'$ for some $\sigma' > \sigma$ and satisfies

$$|h(\rho)| \ll (1 + |\Re \rho|)^{-5-\delta}$$

for some $\delta > 0$ uniformly in the same strip. A contour integration gives

$$2 \sum_{\rho_j^\varphi \in B(\sigma, T)} h(\rho_j^\varphi) - 2 \sum_{\rho_j \in B(\sigma, T)} h(\rho_j) = \frac{1}{2\pi i} \int_{\partial B(\sigma, T)^+} h(\rho) \frac{S'_\varphi(\rho)}{S_\varphi(\rho)} d\rho \quad (3.2)$$

We may rewrite (3.2) as

$$\begin{aligned} & 2h(\rho_0^\varphi) - 2h(0) + 2 \sum_{0 < \rho_j^\varphi < T} h(\rho_j^\varphi) - 2 \sum_{0 < \rho_j < T} h(\rho_j) \\ &= \frac{1}{\pi i} \int_{-i\sigma-T}^{-i\sigma+T} \frac{h(\rho) d\rho}{\rho(\log i\rho - 2\pi c(\varphi))} \\ & \quad - \frac{1}{\pi i} \int_{-i\sigma-T}^{-i\sigma+T} h'(\rho) \log \left(1 - \frac{D(\rho)}{\log i\rho - 2\pi c(\varphi)} \right) d\rho + \partial B(T) \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} \partial B(T) &= \frac{1}{\pi i} \left[h(r) \log \left(\frac{S_\varphi(r)}{\log(ir) - 2\pi c(\varphi)} \right) \right]_{-i\sigma-T}^{-i\sigma+T} \\ &\quad + \frac{1}{\pi i} \int_{-i\sigma+T}^{i\sigma+T} h(\rho) \frac{S'_\varphi}{S_\varphi}(\rho) d\rho. \end{aligned} \tag{3.4}$$

Choose a sequence $\{T_n\}$ away from $\{\rho_j\} \cup \{\rho_j^\varphi\}$ such that $\lim_n T_n = \infty$. By use of the asymptotics (3.1), we show that the integral over the contour $[-i\sigma - T_n, -i\sigma + T_n]$ converges absolutely as $n \rightarrow \infty$. Since Weyl’s law implies that both traces converge absolutely, it follows that $\lim_n \partial B(T_n)$ exists. The main step in the proof of the trace formula is to show that actually $\lim_n \partial B(T_n) = 0$ for a suitable choice of a sequence $\{T_n\}$.

In Lemma 4.1, we construct a sequence $\{T_n\}$ which satisfies

$$|S_\varphi(T_n + iw)| \ll_\epsilon T_n^{4+\epsilon}.$$

We then use this bound together with our knowledge of the existence of $\lim_n \partial B(T_n)$, which holds in particular for a certain test function h_5 with suitable symmetry properties (cf. Lemma 4.2). We exploit the properties of this particular test function to bound $\log |S_\varphi|$ on average on the segments $[-i\sigma + T_n, T_n]$, namely (cf. Lemma 4.3)

$$\left| \int_{-i\sigma+T_n}^{T_n} \log |S_\varphi(\rho)| d\rho \right| \ll T_n^5. \tag{3.5}$$

which allows us to pass to the limit $T_n \rightarrow \infty$ and obtain the trace formula (2.6).

Remark. We are unable to obtain an individual bound on $\log |S_\varphi(\rho)|$ on the segments $[-i\sigma + T_n, T_n]$, but a bound on average suffices for our purposes.

3.2. The Green’s Function on the Torus

The free Green’s function on \mathbb{R}^2 is given by

$$g_\lambda(x, x_0) = \frac{1}{2\pi} K_0(i\rho|x - x_0|), \quad \lambda = \rho^2, \tag{3.6}$$

where K_0 denotes the zeroth Bessel function.

From the integral representation

$$K_0(r) = \int_0^\infty \exp(-r \cosh t) dt \stackrel{w=\cosh t}{=} \int_1^\infty \frac{e^{-wr} dw}{\sqrt{w^2 - 1}}, \quad \Re r > 0 \tag{3.7}$$

we obtain the following integral representation for the free Green’s function on \mathbb{R}^2

$$g_\lambda(x, x_0) = \frac{1}{2\pi} \int_1^\infty \frac{e^{-iw\rho|x-x_0|} dw}{\sqrt{w^2 - 1}}, \quad \lambda = \rho^2, \quad \Im \rho < 0. \tag{3.8}$$

We derive an integral representation for the Green's function on the torus \mathbb{T}^2 by the method of images. Let $\Im\rho < 0$. We have

$$G_\lambda(x, x_0) = \sum_{n \in \mathcal{L}} g_\lambda(x + n, x_0) = \frac{1}{2\pi} \int_1^\infty \frac{k_\rho(w; x, x_0) dw}{\sqrt{w^2 - 1}} \quad (3.9)$$

where

$$k_\rho(w; x, x_0) = \sum_{n \in \mathcal{L}} e^{-iw\rho|x-x_0+n|}. \quad (3.10)$$

Absolute convergence follows from the inequality (note $w \geq 1$)

$$|k_\rho(w; x, x_0)| \leq \sum_{n \in \mathcal{L}} e^{-\sigma w|x-x_0+n|} \ll 1 + \sum_{0 \neq m \in \mathcal{N}} r_{\mathcal{L}}(m) e^{-\sigma\sqrt{m}} \quad (3.11)$$

The Bessel function has the asymptotics

$$K_0(z) = -\log(z/2) - \gamma + o(1), \quad z \rightarrow 0 \quad (3.12)$$

where γ denotes Euler's constant. Therefore, the free Green's function has the asymptotics

$$g_\lambda(x, x_0) = -\frac{1}{2\pi} \log(i\rho|x - x_0|/2) - \frac{\gamma}{2\pi} + o(1) \quad (3.13)$$

as $x \rightarrow x_0$. Thus, we have the following asymptotics for the Green's function on the torus

$$G_\lambda(x, x_0) = -\frac{1}{2\pi} \log(i\rho|x - x_0|/2) - \frac{\gamma}{2\pi} + C_\lambda + o(1) \quad (3.14)$$

as $x \rightarrow x_0$, where

$$C_\lambda = \sum_{n \in \mathcal{L} \setminus \{0\}} g_\lambda(x_0 + n, x_0).$$

3.3.

In view of the spectral expansion of the Green's function G_λ , the spectral function may be written as

$$S_\varphi(\rho) = \lim_{x \rightarrow x_0} \{G_\lambda(x, x_0) - \Re G_1(x, x_0)\} - c_0 \tan \frac{\varphi}{2} \quad (3.15)$$

where

$$c_0 = 1 + \sum_{0 \neq n \in \mathcal{N}} \frac{r_{\mathcal{L}}(n)}{n^2 + 1}.$$

We may rewrite (3.15) as

$$S_\varphi(\rho) = -\frac{1}{2\pi} \log i\rho + \frac{1}{2\pi} \mathcal{D}(\rho) + c(\varphi) \quad (3.16)$$

where

$$k(x) = \sum_{n \in \mathcal{L} \setminus \{0\}} e^{-ix|n|} = \sum_{0 \neq m \in \mathcal{N}} r_{\mathcal{L}}(m) e^{-ix\sqrt{m}} \tag{3.17}$$

$$\mathcal{D}(\rho) = \int_1^{\infty} \frac{k(\rho w) dw}{\sqrt{w^2 - 1}}. \tag{3.18}$$

and

$$c(\varphi) = -\frac{1}{2\pi} \Re \mathcal{D}(-e^{i\pi/4}) - c_0 \tan \frac{\varphi}{2} \tag{3.19}$$

is a real constant.

We have the expression

$$c(\varphi) = c_1 - c_0 \tan \frac{\varphi}{2} \tag{3.20}$$

where

$$c_1 = -\frac{1}{2\pi} \sum_{m \in \mathcal{N}} r_{\mathcal{L}}(m) \int_1^{\infty} \frac{\cos(\sqrt{\frac{m}{2}}w) e^{-\sqrt{\frac{m}{2}}w} dw}{\sqrt{w^2 - 1}}. \tag{3.21}$$

Lemma 3.1. *For sufficiently large $\sigma > 0$ and $\Im \rho = -\sigma$*

$$\frac{|\mathcal{D}(\rho)|}{|\log i\rho - 2\pi c(\varphi)|} < 1. \tag{3.22}$$

Proof. We have

$$|\log i\rho - 2\pi c(\varphi)| \geq |\log \sqrt{\sigma^2 + (\Re \rho)^2} - 2\pi c(\varphi)| \geq \log \sigma - 2\pi |c(\varphi)| \tag{3.23}$$

and

$$|\mathcal{D}(\rho)| \leq \int_1^{\infty} \frac{|k(\rho w)| dw}{\sqrt{w^2 - 1}} \leq \sum_{m \in \mathcal{N}} r_{\mathcal{L}}(m) \int_1^{\infty} \frac{e^{-\sigma\sqrt{m}w} dw}{\sqrt{w^2 - 1}} = f(\sigma) \tag{3.24}$$

which implies for sufficiently large $\sigma > 0$ (in particular it is necessary that $\log \sigma > 2\pi |c(\varphi)|$)

$$\frac{|\mathcal{D}(\rho)|}{|\log i\rho - 2\pi c(\varphi)|} \leq \frac{f(\sigma)}{\log \sigma - 2\pi |c(\varphi)|} < 1. \tag{3.25}$$

□

Let h be an even function, analytic in a strip $|\Im \rho| \leq \sigma_0$, for some $\sigma_0 > \sigma$, which satisfies

$$|h(\rho)| \ll (1 + |\Re \rho|)^{-5-\delta} \tag{3.26}$$

uniformly in the same strip for some $\delta > 0$.

Remark. We restrict ourselves to a smaller space of test functions here to simplify the presentation of our arguments. It is possible to obtain the trace formula for any test function with uniform decay $|h(\rho)| \ll (1 + |\Re \rho|)^{-2-\delta}$.

Let $T > 0$. Define the box

$$B(\sigma, T) = \{\rho \mid |\Re \rho| \leq T, |\Im \rho| \leq \sigma\}.$$

Proposition 3.2. *Denote by $n_j = \rho_j^2, \rho_j \geq 0$, the eigenvalues without counting multiplicities. The new eigenvalues which lie strictly between the n_j are denoted by $\lambda_j^\varphi = (\rho_j^\varphi)^2$. We denote $0 > \lambda_0^\varphi = (\rho_0^\varphi)^2$ where ρ_0^φ is purely imaginary and $\Im \rho_0^\varphi > 0$. Let $\sigma > \Im \rho_0^\varphi$ and $T > 0$ s.t. $T \notin \{\rho_j\}_j \cup \{\rho_j^\varphi\}_j$. We have*

$$\begin{aligned} & 2h(\rho_0^\varphi) - 2h(0) + 2 \sum_{0 < \rho_j^\varphi < T} h(\rho_j^\varphi) - 2 \sum_{0 < \rho_j < T} h(\rho_j) \\ &= \frac{1}{2\pi i} \int_{\partial B(\sigma, T)} h(\rho) \frac{S'_\varphi(\rho)}{S_\varphi(\rho)} d\rho \\ &= \frac{1}{\pi i} \left\{ \int_{-i\sigma-T}^{-i\sigma+T} + \int_{-i\sigma+T}^{i\sigma+T} \right\} h(\rho) \frac{S'_\varphi(\rho)}{S_\varphi(\rho)} d\rho \end{aligned} \quad (3.27)$$

Proof. By contour integration and symmetry. This is clear in view of the spectral expansion

$$S_\varphi(\rho) = -\frac{1}{\rho^2} + \sum_{j=1}^{\infty} r(n_j) \left\{ \frac{1}{n_j - \rho^2} - \frac{n_j}{n_j^2 + 1} \right\} - c_0 \tan \frac{\varphi}{2}. \quad (3.28)$$

□

We may rewrite (3.27) as

$$\begin{aligned} & 2h(\rho_0^\varphi) - 2h(0) + 2 \sum_{0 < \rho_j^\varphi < T} h(\rho_j^\varphi) - 2 \sum_{0 < \rho_j < T} h(\rho_j) \\ &= \frac{1}{\pi i} \int_{-i\sigma-T}^{-i\sigma+T} \frac{h(\rho) d\rho}{\rho(\log i\rho - 2\pi c(\varphi))} \\ & \quad - \frac{1}{\pi i} \int_{-i\sigma-T}^{-i\sigma+T} h'(\rho) \log \left(1 - \frac{\mathcal{D}(\rho)}{\log i\rho - 2\pi c(\varphi)} \right) d\rho + \partial B(T) \end{aligned} \quad (3.29)$$

where

$$\begin{aligned} \partial B(T) &= \frac{1}{\pi i} \left[h(r) \log \left(\frac{S_\varphi(r)}{\log(ir) - 2\pi c(\varphi)} \right) \right]_{-i\sigma-T}^{-i\sigma+T} \\ & \quad + \frac{1}{\pi i} \int_{-i\sigma+T}^{i\sigma+T} h(\rho) \frac{S'_\varphi(\rho)}{S_\varphi(\rho)} d\rho. \end{aligned} \quad (3.30)$$

We have the following fact, analogous to Theorem 12 in [15].

Proposition 3.3. *There exists an increasing sequence $\{t_l\} \subset \mathbb{R}_+ \setminus (\{\rho_j\}_j \cup \{\rho_j^\varphi\}_j)$ such that $\lim_{l \rightarrow \infty} t_l = +\infty$ and*

$$\lim_{l \rightarrow \infty} \partial B(t_l) = 0.$$

Since the sums and integrals in (3.29) (where we take $T = t_l$) converge absolutely as $t_l \rightarrow \infty$, Proposition 3.3, which we will prove in Sect. 4, gives the general trace formula:

Theorem 3.4. *Let h be as (3.26). Let $\sigma > 0$ be large enough s.t. condition (3.22) is satisfied. We have*

$$\begin{aligned} & \sum_{j=0}^{\infty} \{h(\rho_j^\varphi) - h(\rho_j)\} \\ &= \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} \frac{h(\rho) d\rho}{\rho(\log i\rho - 2\pi c(\varphi))} \\ & \quad - \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} h'(\rho) \log \left(1 - \frac{\mathcal{D}(\rho)}{\log i\rho - 2\pi c(\varphi)} \right) d\rho. \end{aligned} \tag{3.31}$$

We call the first term on the RHS of (3.31) the “smooth term”, and the second one the “diffractive term”.

4. Proof of Proposition 3.3

We begin with the following lemma.

Lemma 4.1. *There exists an increasing sequence $\{T_n\} \subset \mathbb{R}_+ \setminus (\{\rho_j\}_j \cup \{\rho_j^\varphi\}_j)$ such that $\lim_{n \rightarrow \infty} T_n = +\infty$ and for $-\sigma \leq w \leq 0$ we have*

$$|S_\varphi(T_n + iw)| \ll_\epsilon T_n^{4+\epsilon}. \tag{4.1}$$

Proof. We can choose an infinite increasing subsequence of Laplacian eigenvalues $\{n_{k(n)}\}_n$ such that $n_{k(n)+1} - n_{k(n)} = \rho_{k(n)+1}^2 - \rho_{k(n)}^2 \gg 1$. This is because the mean spacing between the norms $\{n_j\}$ is of size $\sqrt{\log n_j}$ if the lattice \mathcal{L} is rational and of size 1 if the lattice is irrational. Recall that between two consecutive eigenvalues $n_{k(n)} = \rho_{k(n)}^2$ and $n_{k(n)+1} = \rho_{k(n)+1}^2$ there is exactly one new eigenvalue $\lambda_{k(n)+1}^\varphi = \chi_{k(n)+1}^2$ and $\chi_{k(n)+1} \in (\rho_{k(n)}, \rho_{k(n)+1}) \subset \mathbb{R}_+$ is a zero of the function $S_\varphi(\rho)$, whereas $\rho_{k(n)}, \rho_{k(n)+1}$ are singularities of the same function.

Therefore, we may choose an infinite sequence

$$T_n = \begin{cases} \frac{1}{2}(\rho_{k(n)} + \chi_{k(n)+1}), & \text{if } |\chi_{k(n)+1} - \rho_{k(n)}| \geq |\chi_{k(n)} - \rho_{k(n)+1}| \\ \frac{1}{2}(\rho_{k(n)+1} + \chi_{k(n)+1}), & \text{otherwise.} \end{cases} \tag{4.2}$$

with $|\rho_{k(n)} - \rho_{k(n)+1}| \gg |\rho_{k(n)} + \rho_{k(n)+1}| \asymp T_n^{-1}$. Note in particular that for all $\rho_j \in \mathbb{R}_+$,

$$|\rho_j - T_n| \geq \frac{1}{4} |\rho_{k(n)} - \rho_{k(n)+1}| \gg T_n^{-1}. \quad (4.3)$$

Let $\mu_n(w) = (T_n + iw)^2$, $w \in [-\sigma, 0]$. We have for any $\epsilon > 0$

$$\begin{aligned} |S^\varphi(T_n + iw)| &\ll \sum_{j=0}^{\infty} r_{\mathcal{L}}(n_j) \left| \frac{1}{n_j - \mu_n(w)} - \frac{1}{n_j - i} \right| \\ &\ll \epsilon |i - \mu_N(w)| \sum_{j=0}^{\infty} \frac{n_j^\epsilon}{|n_j - \mu_N(w)||n_j - i|} \end{aligned} \quad (4.4)$$

where we have used the bound $r_{\mathcal{L}}(n) \ll_\epsilon n^\epsilon$. Fix $\alpha \in (\epsilon, 1)$. We split the sum into a central part satisfying $\inf_{w \in [-\sigma, 0]} |n_j - \mu_N(w)| < n_j^\alpha$ and a corresponding tail. For convenience we let $I_n(n_j) = \inf_{w \in [-\sigma, 0]} |n_j - \mu_n(w)|$. The first sum is estimated by

$$\begin{aligned} &\sum_{I_n(n_j) < n_j^\alpha} \frac{n_j^\epsilon}{|n_j - \mu_n(w)||n_j - i|} \\ &\leq \#\{j \mid I_n(n_j) < n_j^\alpha\} \max_{I_n(n_j) < n_j^\alpha} \sup_{w \in [-\sigma, 0]} \left\{ \frac{n_j^\epsilon}{|n_j - \mu_n(w)||n_j - i|} \right\}. \end{aligned} \quad (4.5)$$

Now if $n_j > T_n^2$ then $I_n(n_j) = n_j - T_n^2$. It follows

$$\#\{j \mid I_n(n_j) < n_j^\alpha\} \leq \#\{j \mid n_j \leq T_n^2\} + \#\{j \mid n_j - n_j^\alpha < T_n^2\}. \quad (4.6)$$

Let

$$C(\alpha) = \#\{j \mid n_j \leq 2^{1/(1-\alpha)}\} \quad (4.7)$$

and observe that $n_j > 2^{1/(1-\alpha)}$ implies $n_j^{\alpha-1} < \frac{1}{2}$. So $n_j > 2^{1/(1-\alpha)}$ together with $n_j(1 - n_j^{\alpha-1}) < T_n^2$ implies

$$n_j < 2n_j(1 - n_j^{\alpha-1}) < 2T_n^2. \quad (4.8)$$

Hence

$$\begin{aligned} &\#\{j \mid n_j(1 - n_j^{\alpha-1}) < T_n^2\} \\ &\leq \#\{j \mid n_j \leq 2^{1/(1-\alpha)}, n_j(1 - n_j^{\alpha-1}) < T_n^2\} \\ &\quad + \#\{j \mid n_j > 2^{1/(1-\alpha)}, n_j(1 - n_j^{\alpha-1}) < T_n^2\} \\ &\leq C(\alpha) + \#\{j \mid 2^{1/(1-\alpha)} < n_j < 2T_n^2\} \\ &\ll T_n^2. \end{aligned} \quad (4.9)$$

It follows that

$$\#\{j \mid I_n(n_j) < n_j^\alpha\} \ll T_n^2. \quad (4.10)$$

By the same observations as above, we see that $I(n_j) < n_j^\alpha$ implies $n_j \leq \max\{2^{1/(1-\alpha)}, 2T_n^2\}$. Also for any $j \geq 0$ we have (cf. (4.3))

$$|\rho_j - T_n| \geq \frac{1}{4} |\rho_{k(n)} - \rho_{k(n)+1}| \gg T_n^{-1}$$

which implies

$$\begin{aligned} |n_j - \mu_n(w)| &= |\rho_j^2 - (T_n + iw)^2| = |\rho_j - T_n - iw||\rho_j + T_n + iw| \\ &\geq |\rho_j - T_n|(\rho_j + T_n) \\ &\gg 1. \end{aligned} \tag{4.11}$$

Since $|n_j - i| \geq 1$, we have

$$\max_{I_n(n_j) < n_j^\alpha} \sup_{w \in [-\sigma, 0]} \left\{ \frac{n_j^\epsilon}{|n_j - \mu_n(w)||n_j - i|} \right\} \ll T_n^\epsilon. \tag{4.12}$$

The tail can be bounded as follows

$$\begin{aligned} \sum_{I_n(n_j) \geq n_j^\alpha} \frac{n_j^\epsilon}{|n_j - \mu_n(w)||n_j - i|} &\leq \sum_{I_n(n_j) \geq n_j^\alpha} \frac{n_j^{\epsilon-\alpha}}{|n_j - i|} \\ &\leq \sum_{j=0}^\infty \frac{n_j^{\epsilon-\alpha}}{|n_j - i|} = O(1). \end{aligned} \tag{4.13}$$

Finally note that $|\mu_n(w) - i| \ll T_n^2$. □

Recall

$$\begin{aligned} \partial B(T_n) &= \frac{1}{\pi i} \left[h(r) \log \left(1 - \frac{\mathcal{D}(r)}{\log(ir) - 2\pi c(\varphi)} \right) \right]_{-i\sigma - T_n}^{-i\sigma + T_n} \\ &\quad + \frac{1}{\pi i} \int_{-i\sigma + T_n}^{i\sigma + T_n} h(\rho) \frac{S'_\varphi}{S_\varphi}(\rho) d\rho. \end{aligned} \tag{4.14}$$

We know that $\lim_{n \rightarrow \infty} B(T_n)$ exists (for any test function h which satisfies the uniform bound $|h(\rho)| \ll (1 + |\Re \rho|)^{-2-\delta}$ in the strip $|\Im \rho| \leq \sigma$ —the decay which is required by Weyl’s law to ensure that the trace converges absolutely) and we want to prove that the limit is zero for any test function which satisfies the uniform bound

$$|h(\rho)| \ll (1 + |\Re \rho|)^{-5-\delta} \tag{4.15}$$

in the strip $|\Im \rho| \leq \sigma$.

For the first term we have, in view of $|\mathcal{D}(\rho)| \leq f(\sigma)$ along $\Im \rho = -\sigma$,

$$\left| \log \left(1 - \frac{\mathcal{D}(-i\sigma \pm T_n)}{\log(\sigma \pm iT_n) - 2\pi c(\varphi)} \right) \right| \ll \frac{1}{\log T_n} \tag{4.16}$$

which implies that this term vanishes as $n \rightarrow \infty$.

For the integral, an integration by parts gives

$$\begin{aligned} \int_{-i\sigma+T_n}^{i\sigma+T_n} h(\rho) \frac{S'_\varphi}{S_\varphi}(\rho) d\rho &= [h(\rho) \log S_\varphi(\rho)]_{-i\sigma+T_n}^{i\sigma+T_n} \\ &\quad - \int_{-i\sigma+T_n}^{i\sigma+T_n} h'(\rho) \log S_\varphi(\rho) d\rho. \end{aligned} \quad (4.17)$$

To see that the first term vanishes as $n \rightarrow \infty$, observe that the identity (3.16) and the bound (3.24) imply

$$\begin{aligned} |\log S(\pm i\sigma + T_n)| &= |\log S(-i\sigma \mp T_n)| \\ &\leq |\log |S(\pm i\sigma + T_n)|| + |\arg S(\pm i\sigma + T_n)| \\ &= \log \log T_n + O(1), \end{aligned} \quad (4.18)$$

where we used $|\arg S(\pm i\sigma + T_n)| \ll 1$ as $n \rightarrow \infty$. Similarly we see

$$\int_{-i\sigma+T_n}^{i\sigma+T_n} h'(\rho) \log S_\varphi(\rho) d\rho = \int_{-i\sigma+T_n}^{i\sigma+T_n} h'(\rho) \log |S_\varphi(\rho)| d\rho + O(T_n^{-5}).$$

We have the calculation

$$\begin{aligned} &\int_{T_n}^{i\sigma+T_n} h'(\rho) \log |S_\varphi(\rho)| d\rho \\ &\stackrel{\rho \rightarrow -\rho}{=} - \int_{-T_n}^{-i\sigma-T_n} h'(-\rho) \log |S_\varphi(-\rho)| d\rho \\ &\stackrel{\rho \rightarrow \bar{\rho}}{=} - \int_{T_n}^{-i\sigma+T_n} h'(\bar{\rho}) \log |S_\varphi(\bar{\rho})| d\rho \\ &= \int_{-i\sigma+T_n}^{T_n} h'(\bar{\rho}) \log |S_\varphi(\rho)| d\rho \end{aligned} \quad (4.19)$$

where we used $S_\varphi(\bar{\rho}) = \overline{S_\varphi(\rho)}$, and so the term

$$\int_{-i\sigma+T_n}^{i\sigma+T_n} h'(\rho) \log |S_\varphi(\rho)| d\rho = \int_{-i\sigma+T_n}^{T_n} \{h'(\rho) + h'(\bar{\rho})\} \log |S_\varphi(\rho)| d\rho \quad (4.20)$$

converges to a limit as $n \rightarrow \infty$.

To obtain the result, we require two lemmas. The first lemma constructs an even test function which is analytic in a strip and the real part of whose derivative satisfies a certain polynomial lower bound in T_n on the line segment $[-i\sigma + T_n, T_n]$.

Lemma 4.2. *Choose $\sigma_0 > \sigma$. Let*

$$h_5(\rho) = \frac{-1}{(\rho^2 + \sigma_0^2)^2}.$$

We have for $t \in [-\sigma_0, 0]$ and for all sufficiently large n

$$\Re h'_5(T_n + it) = |\Re h'_5(T_n + it)| \gg \frac{1}{T_n^5} \quad (4.21)$$

as $n \rightarrow \infty$.

Proof. We have

$$h'_5(\rho) = \frac{4\rho}{(\rho^2 + \sigma_0^2)^3}.$$

Let $t \in [-\sigma_0, 0]$. A simple calculation gives

$$\begin{aligned} |\Re h'_5(T_n + it)| &= \left| \Re \left\{ \frac{4(T_n + it)(T_n^2 - t^2 + \sigma_0^2 - 2iT_n t)^3}{((T_n^2 - t^2 + \sigma_0^2)^2 + 4T_n^2 t^2)^3} \right\} \right| \\ &= \Re \left\{ \frac{4(T_n + it)(T_n^2 - t^2 + \sigma_0^2 - 2iT_n t)^3}{((T_n^2 - t^2 + \sigma_0^2)^2 + 4T_n^2 t^2)^3} \right\} \gg \frac{1}{T_n^5} \end{aligned} \quad (4.22)$$

as $n \rightarrow \infty$. □

The second lemma gives a bound on $\log |S_\varphi|$ averaged along the line segment $[-i\sigma + T_n, T_n]$.

Lemma 4.3. *We have the following bound*

$$\left| \int_{-i\sigma + T_n}^{T_n} \log |S_\varphi(\rho)| d\rho \right| \ll T_n^5. \quad (4.23)$$

Proof. We know there exists a constant $c > 0$ such that for all n and $w \in [-\sigma, 0]$ we have

$$|S_\varphi(T_n + iw)| < cT_n^5.$$

In Lemma 4.2, we prove the existence of a test function h_5 which is analytic in the strip $|\Im \rho| \leq \sigma$, satisfies the uniform bound $|h_5(\rho)| \ll (1 + |\Re \rho|)^{-4}$ in this strip and in addition $h_5(\bar{\rho}) = \overline{h_5(\rho)}$ and $\Re h'_5(T_n + iw) = |\Re h'_5(T_n + iw)| \gg T_n^{-5}$. We thus have

$$\begin{aligned}
 & T_n^{-5} \left| \int_{-i\sigma+T_n}^{T_n} \log |S_\varphi(\rho)| d\rho \right| \\
 & \leq T_n^{-5} \int_{-\sigma}^0 -\log(c^{-1}T_n^{-5}|S_\varphi(T_n+iw)|)dw + O(T_n^{-5} \log T_n) \\
 & \ll - \int_{-\sigma}^0 \Re h'_5(T_n+iw) \log(c^{-1}T_n^{-5}|S_\varphi(T_n+iw)|)dw \\
 & \quad + O(T_n^{-5} \log T_n) \\
 & \ll_\epsilon 1
 \end{aligned} \tag{4.24}$$

because $|h_5(\rho)| \ll (1 + |\Re \rho|)^{-4}$ uniformly in $|\Im \rho| \leq \sigma$ and therefore

$$\lim_{n \rightarrow \infty} \int_{-i\sigma+T_n}^{T_n} \Re h'_5(\rho) \log |S_\varphi(\rho)| d\rho$$

exists. □

We obtain

$$\left| \int_{-i\sigma+T_n}^{i\sigma+T_n} h'(\rho) \log |S_\varphi(\rho)| d\rho \right| \ll T_n^{-\delta}$$

in view of the identity (4.20). We also used that by Cauchy's theorem the analyticity and decay of h in $|\Im \rho| \leq \sigma_0$, where $\sigma_0 > \sigma$, imply the analyticity of h' in $|\Im \rho| \leq \sigma$ and the uniform decay

$$|h'(\rho)| \ll (1 + |\Re \rho|)^{-5-\delta}$$

in the same strip. It follows that

$$\lim_{n \rightarrow \infty} \partial B(T_n) = 0$$

which proves Proposition 3.3.

5. Proof of Proposition 2.1

5.1.

We want to apply the trace formula in order to obtain information about the average spacing between new eigenvalues and old eigenvalues. Let $h(\rho) = e^{-\beta\rho^2}$, for small $\beta > 0$. Upon dividing through by β we can rewrite the l.h.s. of the trace formula (3.31) as

$$\begin{aligned} \frac{1}{\beta} \sum_{j=0}^{\infty} \{e^{-\beta\lambda_j^\varphi} - e^{-\beta n_j}\} &= \sum_{j=0}^{\infty} e^{-\beta\lambda_j^\varphi} \frac{1 - e^{-\beta(n_j - \lambda_j^\varphi)}}{\beta} \\ &= \sum_{j=0}^{\infty} d_j e^{-\beta\lambda_j^\varphi} + O(\beta^{-1/2}) \end{aligned} \tag{5.1}$$

where $d_j = n_j - \lambda_j^\varphi > 0$. The last line follows from the following lemma.

Lemma 5.1. *We have the bound*

$$\sum_{j=0}^{\infty} d_j e^{-\beta\lambda_j^\varphi} \left(1 - \frac{1 - e^{-\beta d_j}}{\beta d_j}\right) \ll \beta^{-1/2} \tag{5.2}$$

Proof. For $x > 0$, we have the inequality

$$0 < 1 - \frac{1 - e^{-x}}{x} < x$$

and the bound $d_j \ll n_j^{1/4}$ for $j \geq 1$ (cf. the greedy algorithm in [7], p. 7). It follows

$$\begin{aligned} \sum_{j=0}^{\infty} d_j e^{-\beta\lambda_j^\varphi} \left(1 - \frac{1 - e^{-\beta d_j}}{\beta d_j}\right) &< \beta \sum_{j=0}^{\infty} d_j^2 e^{-\beta\lambda_j^\varphi} \\ &\ll \beta \sum_{j=1}^{\infty} n_j^{1/2} e^{-\beta\lambda_j^\varphi} + \beta e^{-\beta\lambda_0^\varphi} \\ &< \beta \sum_{j=0}^{\infty} n_{j+1}^{1/2} e^{-\beta n_j} + \beta e^{-\beta\lambda_0^\varphi} \end{aligned} \tag{5.3}$$

and the bound $N_\varphi(x) \ll x$ permits us to bound the sum by the following integral:

$$\beta \sum_{j=0}^{\infty} n_{j+1}^{1/2} e^{-\beta n_j} \ll \beta \int_0^\infty x^{1/2} e^{-\beta x} dx \ll \beta^{-1/2}. \tag{5.4}$$

□

5.2. The Smooth Term

We have the following bound on the smooth term.

Proposition 5.2. *As $\beta \searrow 0$*

$$\frac{1}{2\pi} \left| \int_{-i\sigma-\infty}^{-i\sigma+\infty} \frac{e^{-\beta\rho^2} d\rho}{\rho(\log(i\rho) - 2\pi c(\varphi))} \right| \ll \frac{1}{\log \frac{1}{\beta}}. \tag{5.5}$$

Proof. Denote by C_δ the contour following a semicircle in the lower halfplane centered at the origin of radius δ , where $e^{2\pi c(\varphi)} > \delta > 0$, starting from $-\delta$ and

finishing at δ . By shifting the contour across the pole at $\rho = -ie^{2\pi c(\varphi)}$ to the real line we obtain

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} \frac{e^{-\beta\rho^2} d\rho}{\rho \log(i\rho e^{-2\pi c(\varphi)})} \\ &= e^{\beta e^{4\pi c(\varphi)}} + \frac{1}{2\pi i} \left\{ \int_{C_\delta} + \int_{\mathbb{R} \setminus (-\delta, \delta)} \right\} \frac{e^{-\beta\rho^2} d\rho}{\rho \log(i\rho e^{-2\pi c(\varphi)})}. \end{aligned} \quad (5.6)$$

Note that the integral over the semicircle vanishes as $\delta \rightarrow 0$.

We may pick the branch of the complex logarithm in such a way that $\arg(x) = \pi/2$ if $x < 0$ and $\arg(x) = 3\pi/2$ if $x > 0$. Then for real $\rho \neq 0$

$$\begin{aligned} \frac{1}{\log(i\rho e^{-2\pi c(\varphi)})} &= \frac{1}{\log(|\rho|e^{-2\pi c(\varphi)} + i(\frac{\pi}{2} + \arg(\rho e^{-2\pi c(\varphi)})))} \\ &= \frac{\log(|\rho|e^{-2\pi c(\varphi)} - i(\frac{\pi}{2} + \arg(\rho e^{-2\pi c(\varphi)})))}{\log^2(|\rho|e^{-2\pi c(\varphi)} + \pi^2/4)} \end{aligned} \quad (5.7)$$

and it follows that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\mathbb{R} \setminus (-\delta, \delta)} \frac{e^{-\beta\rho^2} d\rho}{\rho \log(i\rho e^{-2\pi c(\varphi)})} \\ &= -\frac{1}{2\pi} \int_{\mathbb{R} \setminus (-\delta, \delta)} e^{-\beta\rho^2} \frac{\frac{\pi}{2} + \arg(\rho e^{-2\pi c(\varphi)})}{\rho(\log^2(|\rho|e^{-2\pi c(\varphi)} + \frac{\pi^2}{4}))} d\rho \\ &= -\frac{1}{2} \int_{\delta}^{\infty} \frac{e^{-\beta\rho^2} d\rho}{\rho(\log^2(\rho e^{-2\pi c(\varphi)} + \frac{\pi^2}{4}))} \\ &= -\frac{1}{2} \int_{e^{-2\pi c(\varphi)}\delta}^{\infty} \frac{e^{-\beta e^{4\pi c(\varphi)} r^2} dr}{r(\log^2 r + \frac{\pi^2}{4})} \\ &= -\frac{1}{2} \int_{-2\pi c(\varphi) + \log \delta}^{\infty} \frac{e^{-\beta e^{4\pi c(\varphi)} e^{2t}} dt}{t^2 + \frac{\pi^2}{4}} \\ &\rightarrow -\frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-\beta e^{4\pi c(\varphi)} e^{2t}} dt}{t^2 + \frac{\pi^2}{4}} \quad \text{as } \delta \rightarrow 0. \end{aligned} \quad (5.8)$$

Since

$$\int_{-\infty}^{\infty} \frac{dt}{t^2 + \frac{\pi^2}{4}} = 2,$$

we obtain in view of (5.6)

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} \frac{e^{-\beta\rho^2} d\rho}{\rho \log(i\rho e^{-2\pi c(\varphi)})} \\ &= e^{\beta e^{4\pi c(\varphi)}} - 1 + \frac{1}{2} \int_{-\infty}^{\infty} \frac{1 - e^{-\beta e^{4\pi c(\varphi)} e^{2t}}}{t^2 + \frac{\pi^2}{4}} dt. \end{aligned} \tag{5.9}$$

Let $\gamma = e^{4\pi c(\varphi)}\beta$. We proceed by dividing the integral on the r.h.s. into two integrals over the ranges $(-\infty, \frac{1-\epsilon}{2}|\log \gamma|)$ and $[\frac{1-\epsilon}{2}|\log \gamma|, \infty)$ for some small $\epsilon > 0$.

We then bound the first integral as follows

$$\int_{-\infty}^{\frac{1-\epsilon}{2}|\log \gamma|} \frac{1 - e^{-\gamma e^{2t}}}{t^2 + \frac{\pi^2}{4}} dt \ll \gamma \int_{-\infty}^{\frac{1-\epsilon}{2}|\log \gamma|} e^{2t} dt = \frac{1}{2}\gamma^\epsilon \tag{5.10}$$

where we note that

$$|1 - e^{-\gamma e^{2t}}| \ll \gamma e^{2t},$$

because $t < \frac{1-\epsilon}{2}|\log \gamma|$ implies $\gamma e^{2t} < \gamma^\epsilon$.

We bound the second integral by

$$\frac{1}{2} \int_{\frac{1-\epsilon}{2}|\log \gamma|}^{\infty} \frac{|1 - e^{-\gamma e^{2t}}| dt}{t^2 + \frac{\pi^2}{4}} < \int_{\frac{1-\epsilon}{2}|\log \gamma|}^{\infty} t^{-2} dt = \frac{2}{(1-\epsilon)|\log \gamma|}. \tag{5.11}$$

□

5.3. The Diffractive Term

We continue with the bound on the diffractive term.

Proposition 5.3. *Let $h(\rho) = e^{-\beta\rho^2}$. As $\beta \searrow 0$*

$$\left| \int_{-i\sigma-\infty}^{-i\sigma+\infty} h'(\rho) \log \left(1 - \frac{\mathcal{D}(\rho)}{\log i\rho} \right) d\rho \right| \ll \frac{1}{\log \frac{1}{\beta}}. \tag{5.12}$$

Proof. (3.22) allows us to estimate

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} h'(-i\sigma + s) \log \left(1 - \frac{\mathcal{D}(-i\sigma + s)}{\log i(-i\sigma + s) - 2\pi c(\varphi)} \right) ds \right| \\ & \ll \int_{-\infty}^{\infty} |h'(-i\sigma + s)| \frac{|\mathcal{D}(-i\sigma + s)|}{|\log i(-i\sigma + s) - 2\pi c(\varphi)|} ds. \end{aligned} \tag{5.13}$$

We have

$$|h'(-i\sigma + s)| = 2\beta|-i\sigma + s|e^{-\beta(-i\sigma+s)^2} \leq 2\beta(\sigma + |s|)e^{\beta\sigma^2 - \beta s^2} \tag{5.14}$$

and

$$|\mathcal{D}(-i\sigma + s)| \leq \sum_{m \in \mathcal{N}} r_{\mathcal{L}}(m) \int_1^{\infty} \frac{e^{-\sigma\sqrt{m}w} dw}{\sqrt{w^2 - 1}} = f(\sigma) \quad (5.15)$$

and finally

$$|\log(\sigma + is)| \geq \frac{1}{2} \log(\sigma^2 + s^2). \quad (5.16)$$

We continue our estimate as follows (recall $\sigma > \max\{1, e^{2\pi c(\varphi)}\}$)

$$\begin{aligned} & \int_{-\infty}^{\infty} |h'(-i\sigma + s)| \frac{|\mathcal{D}(-i\sigma + s)|}{|\log i(-i\sigma + s) - 2\pi c(\varphi)|} ds \\ & \leq 8\beta e^{\beta\sigma^2} f(\sigma) \int_0^{\infty} \frac{(\sigma + s)e^{-\beta s^2} ds}{\log(\sigma^2 + s^2) - 4\pi c(\varphi)} \end{aligned} \quad (5.17)$$

and the integral is bounded by

$$\begin{aligned} \int_0^{\infty} \frac{(\sigma + s)e^{-\beta s^2} ds}{\log(\sigma^2 + s^2) - 4\pi c(\varphi)} & < \frac{\sigma}{(2 \log \sigma - 4\pi c(\varphi))\beta^{1/2}} \int_0^{\infty} e^{-w^2} dw \\ & + \int_0^{\infty} \frac{se^{-\beta s^2} ds}{\log(\sigma^2 + s^2) - 4\pi c(\varphi)}. \end{aligned} \quad (5.18)$$

Let $\gamma = e^{4\pi c(\varphi)}\beta$ and $\xi = e^{-2\pi c(\varphi)}\sigma$. We bound the second integral on the r.h.s. as follows

$$\begin{aligned} \int_0^{\infty} \frac{se^{-\beta s^2} ds}{\log(\sigma^2 + s^2) - 4\pi c(\varphi)} & = \frac{1}{\beta} \int_0^{\infty} \frac{we^{-w^2} ds}{\log(\sigma^2 + \frac{w^2}{\beta}) - 4\pi c(\varphi)} \\ & = \frac{1}{\beta} \int_0^{\infty} \frac{we^{-w^2} ds}{\log(\xi^2 + \frac{w^2}{\gamma})} \\ & \ll \frac{1}{\beta \log \xi^2} \int_0^{\gamma^\epsilon} we^{-w^2} dw \\ & + \frac{1}{\beta(1 - 2\epsilon) \log \frac{1}{\gamma}} \int_{\gamma^\epsilon}^{\infty} we^{-w^2} dw \end{aligned} \quad (5.19)$$

for small $\epsilon > 0$. The last line follows, since for $w \geq \gamma^\epsilon$ we have (assuming $\gamma < 1$)

$$\begin{aligned} 1 + \frac{\log(\gamma\sigma^2 + w^2)}{\log \frac{1}{\gamma}} &\geq 1 + \frac{2 \log w + \log(1 + \gamma\sigma^2/w^2)}{\log(\frac{1}{\gamma})} \\ &\geq 1 - 2\epsilon + O\left(\frac{\gamma^{1-2\epsilon}}{\log \frac{1}{\gamma}}\right). \end{aligned} \quad (5.20)$$

Since

$$\int_0^{\gamma^\epsilon} w e^{-w^2} dw = O(\gamma^{2\epsilon})$$

we have

$$\int_{-\infty}^{\infty} |h'(-i\sigma + s)| \frac{|\mathcal{D}(-i\sigma + s)|}{|\log i(-i\sigma + s)|} ds \ll \frac{1}{\log(\frac{1}{\gamma})} + O(\gamma^{2\epsilon}).$$

□

6. The Three-Dimensional Case

The three-dimensional case is very different. Consider the three-dimensional flat torus $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathcal{L}_0^3$, where $\mathcal{L}_0^3 = \mathbb{Z}(1/ab, 0, 0) \oplus \mathbb{Z}(0, a, 0) \oplus \mathbb{Z}(0, 0, b)$ for some $a, b > 0$. Denote by \mathcal{L}^3 the dual lattice of \mathcal{L}_0^3 . The eigenvalues of the Laplacian on \mathbb{T}^3 are the norms of the vectors of the dual lattice \mathcal{L}^3 . We denote the set of norms of the dual lattice vectors by \mathcal{N}^3 and the multiplicity of an eigenvalue $n \in \mathcal{N}^3$ is denoted by

$$r_3(n) = \#\{\xi \in \mathcal{L}^3 : |\xi|^2 = n\}. \quad (6.1)$$

Let η_j^φ be the perturbed eigenvalues of the point scatterer on \mathbb{T}^3 and η_j the unperturbed eigenvalues counted without multiplicity (the norms). The ordering is

$$\eta_0^\varphi < 0 = \eta_0 < \eta_1^\varphi < \eta_1 < \cdots < \eta_j^\varphi < \eta_j. \quad (6.2)$$

Our main result is the following. Let $d_j := \eta_j - \eta_j^\varphi > 0$.

Proposition 6.1. *We have as $\beta \searrow 0$*

$$\sum_{j=0}^{\infty} d_j e^{-\beta \eta_j^\varphi} = \frac{1}{2\beta} + O(\beta^{-3/4}). \quad (6.3)$$

6.1. The Green's Function in Three Dimensions

The free Green's function on \mathbb{R}^3 is given by (cf. [16], p. 842, eq. (3.4))

$$g_\eta(x, x_0) = \frac{e^{-i\rho|x-x_0|}}{4\pi|x-x_0|}, \quad \Im\rho < 0, \quad \rho^2 = \eta. \quad (6.4)$$

We periodise to obtain the Green's function on \mathbb{T}^3 .

$$G_\eta(x, x_0) = \frac{1}{4\pi} \sum_{n \in \mathcal{L}^3} \frac{e^{-i\rho|x-x_0+n|}}{|x-x_0+n|} \quad (6.5)$$

In particular, the deficiency elements are given by

$$G_{\pm i}(x, x_0) = \frac{1}{4\pi} \sum_{n \in \mathcal{L}^3} \frac{e^{-\frac{|x-x_0+n|}{\sqrt{2}}} \cos\left(\frac{|x-x_0+n|}{\sqrt{2}}\right)}{|x-x_0+n|} \quad (6.6)$$

where we note that $\pm i = \left(\frac{\pm 1 - i}{\sqrt{2}}\right)^2$.

The spectral function is given by

$$S_3^\varphi(\rho) = \lim_{x \rightarrow x_0} (G_\lambda - \Re G_i)(x, x_0) - \tan \frac{\varphi}{2} = \frac{-i\rho}{4\pi} + D_3^\varphi(\rho) \quad (6.7)$$

where

$$D_3^\varphi(\rho) = -\tan \frac{\varphi}{2} + \frac{1}{4\pi\sqrt{2}} + \frac{1}{4\pi} \sum_{n \in \mathcal{N}^3} r_3(n) \frac{e^{-i\rho n} - e^{-\frac{n}{\sqrt{2}}} \cos\left(\frac{n}{\sqrt{2}}\right)}{n} \quad (6.8)$$

where the first and second terms in (6.7) and (6.8) come from the regularisation (where $h = x - x_0$)

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{e^{-i\rho|h|} - e^{-\frac{|h+n|}{\sqrt{2}}} \cos\left(\frac{|h+n|}{\sqrt{2}}\right)}{4\pi|h|} \\ &= \lim_{h \rightarrow 0} \frac{1 - i\rho|h| + O(|h|^2) - (1 - \frac{|h|}{\sqrt{2}})(1 + O(|h|^2))}{4\pi|h|} \\ &= \frac{-i\rho}{4\pi} + \frac{1}{4\pi\sqrt{2}}. \end{aligned} \quad (6.9)$$

6.2. The Trace Formula in Three Dimensions

We require the following lemma.

Lemma 6.2. *For sufficiently large $\sigma > |\Im\rho_0^\varphi|$, we have for $\Im\rho = -\sigma$*

$$\frac{4\pi|D_3^\varphi(\rho)|}{|\rho|} < 1. \quad (6.10)$$

Proof. For $\Im\rho = -\sigma$, it can easily be seen from (6.8) that $|D_3^\varphi(\rho)| = O(1)$. Furthermore $|\rho| \geq \sigma$. So (6.10) certainly holds for sufficiently large σ . \square

We have the following trace formula for a point scatterer in three dimensions.

Theorem 6.3. *Let h be as above and $\sigma > |\Im \rho_0^\varphi|$ large enough such that (6.10) is satisfied. Then we have*

$$\begin{aligned} & \sum_{j=0}^{\infty} \{h(\rho_j^\varphi) - h(\rho_j)\} \\ &= \frac{1}{2}h(0) + \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} h'(\rho) \log \left(1 + \frac{4\pi i D_3^\varphi(\rho)}{\rho} \right) d\rho \end{aligned} \quad (6.11)$$

Proof. Following the argument in the proof of the trace formula for two dimensions we obtain for $\sigma > |\Im \rho_0^\varphi|$ the trace formula (an analogue of Krein’s famous trace formula)

$$\sum_{j=0}^{\infty} \{h(\rho_j^\varphi) - h(\rho_j)\} = \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} h'(\rho) \log S_3(\rho) d\rho. \quad (6.12)$$

In view of (6.7), we rewrite the r.h.s. as

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} h'(\rho) \log S_3(\rho) d\rho \\ &= \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} h'(\rho) \log \left(\frac{-i\rho}{4\pi} \right) d\rho \\ & \quad + \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} h'(\rho) \log \left(1 + \frac{4\pi i D_3^\varphi(\rho)}{\rho} \right) d\rho \end{aligned} \quad (6.13)$$

and the first term can be evaluated by integration by parts and shifting the contour

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} h'(\rho) \log \left(\frac{-i\rho}{4\pi} \right) d\rho \\ &= \frac{1}{2\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} \frac{h(\rho) d\rho}{\rho} \\ &= \frac{1}{2\pi i} \int_{C_\delta} \frac{h(\rho) d\rho}{\rho} + \underbrace{\frac{1}{2\pi i} \int_{\mathbb{R} \setminus (-\delta, \delta)} \frac{h(\rho) d\rho}{\rho}}_{=0} \\ &= \frac{1}{2}h(0) \end{aligned} \quad (6.14)$$

where we recall that for some small δ the contour C_δ denotes the lower semi-circle connecting $-\delta$ and δ on the real line. □

6.3. Proof of Proposition 6.1

Let $h(\rho) = e^{-\beta\rho^2}$. In this case, the trace formula gives us

$$\begin{aligned} & \sum_{j=0}^{\infty} \{e^{-\beta\eta_j^\varphi} - e^{-\beta\eta_j}\} \\ &= \frac{1}{2} - \frac{\beta}{\pi i} \int_{-i\sigma-\infty}^{-i\sigma+\infty} \rho e^{-\beta\rho^2} \log\left(1 + \frac{4\pi i D_3^\varphi(\rho)}{\rho}\right) d\rho \end{aligned} \quad (6.15)$$

and in view of (6.10) we have for $\Im\rho = -\sigma$ the bound

$$\left| \log\left(1 + \frac{4\pi i D_3^\varphi(\rho)}{\rho}\right) \right| \ll \frac{|D_3^\varphi(\rho)|}{|\rho|} \quad (6.16)$$

which implies

$$\begin{aligned} & \int_{-i\sigma-\infty}^{-i\sigma+\infty} |\rho| |e^{-\beta\rho^2}| \left| \log\left(1 + \frac{4\pi i D_3^\varphi(\rho)}{\rho}\right) \right| |d\rho| \\ & \ll \int_{-i\sigma-\infty}^{-i\sigma+\infty} |e^{-\beta\rho^2}| |D_3^\varphi(\rho)| |d\rho| \\ & \ll e^{\beta\sigma^2} \int_{-\infty}^{\infty} e^{-\beta t^2} dt = O\left(\frac{1}{\sqrt{\beta}}\right) \end{aligned} \quad (6.17)$$

and therefore

$$\frac{1}{\beta} \sum_{j=0}^{\infty} \{e^{-\beta\eta_j^\varphi} - e^{-\beta\eta_j}\} = \frac{1}{2\beta} + O\left(\frac{1}{\sqrt{\beta}}\right). \quad (6.18)$$

An analogue of the greedy algorithm is required for the proof of Proposition 6.1. We state this as a lemma.

Lemma 6.4. *Let $d_j := \eta_j - \eta_j^\varphi$. We have the bound*

$$d_j \ll \eta_j^{1/8}. \quad (6.19)$$

Proof. Recall that each η_j is of the form $q(m, n, k) = am^2 + bn^2 + ck^2$ for real numbers $a, b, c > 0$ and integers m, n, k . We need to show that for each j , we can pick m, n, k such that

$$|\eta_j^\varphi - q(m, n, k)| \ll \eta_j^{1/8}$$

where the implied constant does not depend on our choice. Let $s_1 = \eta_j^\varphi - am^2$. Let $m = \lfloor \sqrt{\eta_j^\varphi/a} \rfloor$ and observe that $s_1 = \eta_j^\varphi - am^2 \ll (\eta_j^\varphi)^{1/2} < \eta_j^{1/2}$. Now let $s_2 = s_1 - bn^2$ and choose $n = \lfloor \sqrt{s_1/b} \rfloor$ so that $s_2 = s_1 - bn^2 \ll s_1^{1/2}$. Finally

choose $k = \lfloor \sqrt{s_2/c} \rfloor$ such that $s_2 - ck^2 \ll s_2^{1/2}$. With the above choices of m, n, k we have

$$|\eta_j^\varphi - q(m, n, k)| = s_2 - ck^2 \ll s_2^{1/2} \ll s_1^{1/4} \ll \eta_j^{1/8}.$$

□

The following lemma implies Proposition 6.1.

Lemma 6.5. *We have the following identity*

$$\sum_{j=0}^{\infty} d_j e^{-\beta \eta_j^\varphi} = \frac{1}{\beta} \sum_{j=0}^{\infty} \{e^{-\beta \eta_j^\varphi} - e^{-\beta \eta_j}\} + O(\beta^{-3/4}) \tag{6.20}$$

Proof. It is sufficient to prove the bound

$$\sum_{j=0}^{\infty} d_j e^{-\beta \lambda_j^\varphi} \left(1 - \frac{1 - e^{-\beta d_j}}{\beta d_j}\right) \ll \beta^{-3/4}. \tag{6.21}$$

For $x > 0$, we have the inequality

$$0 < 1 - \frac{1 - e^{-x}}{x} < x.$$

It follows from the inequality and Lemma 6.4

$$\begin{aligned} \sum_{j=0}^{\infty} d_j e^{-\beta \eta_j^\varphi} \left(1 - \frac{1 - e^{-\beta d_j}}{\beta d_j}\right) &< \beta \sum_{j=0}^{\infty} d_j^2 e^{-\beta \eta_j^\varphi} \\ &\ll \beta \sum_{j=1}^{\infty} \eta_j^{1/4} e^{-\beta \eta_j^\varphi} + \beta e^{-\beta \eta_0^\varphi} \\ &< \beta \sum_{j=0}^{\infty} \eta_{j+1}^{1/4} e^{-\beta \eta_j} + \beta e^{-\beta \eta_0^\varphi} \end{aligned} \tag{6.22}$$

and the bound $N_\varphi(x) \ll x^{3/2}$ permits us to estimate the sum by the following integral:

$$\beta \sum_{j=0}^{\infty} \eta_{j+1}^{1/4} e^{-\beta \eta_j} \ll \beta \int_0^\infty x^{1/4} e^{-\beta x} x^{1/2} dx \ll \beta^{-3/4}. \tag{6.23}$$

□

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